Estimating the terms of the identities with the help of the inequalities $0 \leqslant \varphi \leqslant \varphi_{m0} < 1$, $\varphi^2 (1-\varphi)^{-1} < (1-\varphi_{m0})^{-1}$, we obtain the inequality (2.4).

Thus we have shown that in case of the motion of a viscous compressible fluid with the Tate equation of state, a convex velocity profile forms at high densities between the plates or in a pipe of circular cross-section, whose amplitude increases exponentially in the downstream direction. The velocity has the form $u = \bar{f}\varphi$, where $\bar{f} = AL(1+\Delta p/A)^{x/L}[(\eta_v + {}^1/_3\eta_s)\ln(1+\Delta p/A)]^{-1}$, and

$$\varphi = \varphi(y/a)\ (j=1),\quad \varphi = \varphi(r/a)\ (j=2)$$

is a dimensionless function taking values in the interval $[0, 1[$.

A quantitative estimate of the influence of the volume viscosity on the fluid flow was obtained by solving (2.3) numerically for the case when $j=1$, $A\rho_0 a^2\eta_s^{-2} = 1, aL^{-1}\ln(1+\Delta p/A) = 0.04$, and for various values of η_v/η_s. The results obtained were used to draw a graph of $\bar{u} = (\eta_v/\eta_s + {}^1/_3)^{-1}\varphi_{\max}$ versus $\lg \eta_v/\eta_s$ Fig.4. The function differs from the maximum velocity by a dimensional multiplier independent of η_v. In addition, graphs were drawn of the function $\varphi(y/a)$, characterizing the flow velocity profile for various values of η_v/η_s (Fig.5) where the curves 1, 2, 3 correspond to the values $\eta_v/\eta_s = 100, 400, 1000$. Figs.4 and 5 illustrate the assertion proved in Sect.2 that then the volume viscosity increases and other parameters are kept constant, $\varphi_{\max}$ tends to unity and the relation $u_{\max} \approx \bar{f}$ is satisfied asymptotically for the maximum velocity of flow.

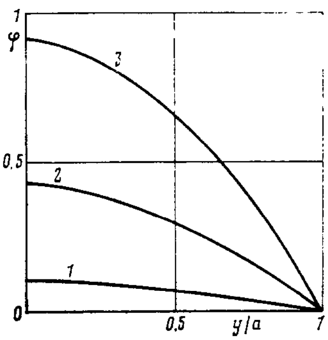

Fig.5

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SHOCK WAVES IN AN ISOTHERMAL GAS IN THE PRESENCE OF REACTION FORCES*

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One-dimensional isothermal gas flow taking into account reaction forces which depend linearly on the velocity is considered. Problems of gas flow with and without convective terms are formulated. Their analytic and numerical solutions are obtained, and the possibility of obtaining shock waves reflected within the medium is indicated.

The flows in question arise when a gas is filtered through porous media, during its passage along pipes and major cracks, when porous bodies move in gaseous media, and in a number of technological processes /1, 2/. A system of equations describing the motion of a gas taking frictional forces into account if given in /3, 4/. The general types of systems of quasilinear equations were studied in /5/.

1. Formulation of the problem. A system of equations describing a one-dimensional isothermal gas flow with resistance forces linear with respect to the velocities, has the form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1.1}$$

$$\rho\left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u\right) = -\frac{\partial}{\partial x}P - au, \quad P = c^2\rho$$

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Here ρ is the density, P is the pressure, u is the gas flow velocity, c is the isothermal speed of sound, and a is constant coefficient of resistance.

As a rule the convective terms are omitted from the second equation of (1.1), since it seems that large resistance forces would cause them to become, after a time $t \sim \rho/a$, vanishingly small compared with the remaining terms. It appears however that under certain initial and boundary conditions the solutions of (1.1) with and without convective terms differ from each other quantitatively, as well as qualitatively.

We take the initial and boundary conditions in the form ((A and u_0 are constants)

$$\rho(x; t=0) = 0, \ u(x; t=0) = 0 \tag{1.2}$$

$$\rho(x=0; t) = At, \ u(x=0; t) = u_0 \tag{1.3}$$

We will seek a selfsimilar solution to problem (1.1)—(1.3)

$$\rho = Atf(\theta), \ u = c\varphi(\theta); \ \theta = x/(ct) \tag{1.4}$$

Here f and φ are dimensionless analogues of the density and velocity of motion of the gas, and θ is the selfsimilar variable. Using the variables (1.4), we can write problem (1.1)—(1.3) in the form

$$f'(\varphi - \theta) + \varphi' f = -f. \tag{1.5}$$

$$f' - \varphi' f(\theta - \varphi) = -\sigma\varphi; \ \sigma = a/A$$

$$f(\theta \to \infty) = 0 \tag{1.6}$$

$$f(\theta = 0) = 1, \ \varphi(\theta = 0) = \varphi_0; \ \varphi_0 = u_0/c \tag{1.7}$$

When the convective terms are neglected, the second equation of (1.5) is replaced by $f' = -\sigma\varphi$ /6—8/.

The simple wave /6/

$$\varphi = \sigma^{-1/2}, \ f = 1 - \sigma^{1/2}\theta, \ \theta \leqslant \theta_0 \tag{1.8}$$

$$\varphi = 0, \ f = 0, \ \theta > \theta_0; \ \theta_0 = \sigma^{-1/2}$$

represents the solution of system (1.5) with conditions (1.6) and the first condition of (1.7). We see that in this case the rate of gas inflow $\varphi(\theta = 0)$ depends on the resistance coefficient a and the constant A.

When the rate of gas inflow is greater than the speed of sound $(\varphi(\theta = 0) > 1)$, two boundary conditions (1.7) must be given for system (1.5).

The solution of problem (1.5)—(1.7) when $\sigma = 0$ (without the reaction force) has the form

$$\varphi - \varphi_0 = \frac{1}{2\sqrt{2}} \ln\left[\frac{\varphi - \theta + \sqrt{2}}{\varphi_0 + \sqrt{2}} \, \frac{\varphi_0 - \sqrt{2}}{\varphi - \theta - \sqrt{2}}\right] \tag{1.9}$$

$$f = \sqrt{\frac{(\varphi - \theta)^2 - 2}{\varphi_0^2 - 2}}, \quad 1 < \varphi_0 < \sqrt{2}, \quad \sqrt{2} < \varphi_0$$

$$\varphi = \theta + \sqrt{2}, \ f = \exp\{-\sqrt{2}\theta\}, \ \varphi_0 = \sqrt{2}$$

Fig.1 shows the relationship between the gas pressure $f(\theta)$ and its velocity $\varphi(\theta)$ for $\varphi_0 = 1.2;\ 1.4;\ 2;\ 5$ (curves 1—4). Qualitative differences in the relationships $f(\theta)$ and $\varphi(\theta)$ when $\varphi_0 > \varphi_0^* = \sqrt{2}\,(\varphi'' > 0)$ and $\varphi_0 < \varphi_0^*\,(\varphi'' < 0)$, are connected with the boundary condition $\rho(0; t) = At$. When $\rho(0; t) = At^{\alpha}$, the value of $\varphi_0^* \neq \sqrt{2}$ depends on α.

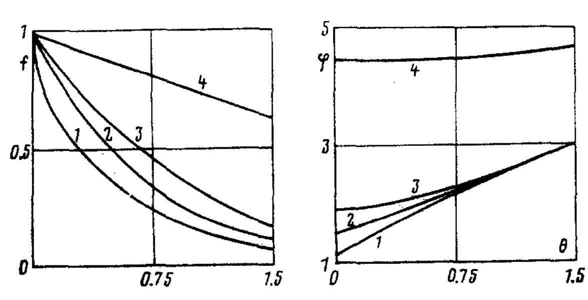

Fig.1

From solution (1.9) it follows that when $\sigma = 0$ and $\varphi_0 > 1$, $\varphi \to \infty$, as θ increases, i.e. the velocity of propagation of the gas front is infinite.

2. Solution of the system of equations for $\sigma > 0$.

From the second equation of (1.5) it follows that when $\sigma > 0$, the gas propagates with a finite velocity. Let θ_1 be the selfsimilar coordinate of the gas expansion front. Since the velocity of motion of the gas at the front is the same as the velocity of motion of the front itself, from (1.4) we obtain

$$\varphi(\theta = \theta_1) = \varphi_1 = \theta_1 \tag{2.1}$$

From (1.5) we see that $\theta < \theta_1$ when $\varphi > 0$.

Let us now consider the motion of a gas when $\varphi_0 > 1$. The characteristics corresponding to $\xi_2 = u - c$, emerge from the straight lines $x = 0$ and $x = v_1 t$ where $v_1 = c\varphi_1$. Since Eqs. (1.1) are not linear, the behaviour of the characteristics depends on the solutions. Two cases are possible: either the characteristics corresponding to ξ_2 and emerging from the straight lines $x = 0$ and $x = v_1 t$, intersect on some line (a straight line, since the problem is selfsimilar), or the field of characteristics is continuous, whereupon a characteristic emerges from the point $x = 0, t = 0$ The space occupied by the gas is divided in the first case by the line at which the characteristics intersect, and in the second case by the characteristic emerging from $x = 0, t = 0$.

Fig.2 shows the possible fields of characteristics, and the straight line $X(t)$ is shown by a dashed line. In region *1* the solution is determined by the boundary conditions only (the characteristics corresponding to $\xi_1 = u + c$ and $\xi_2 = u - c$, emerge from the straight line $x = 0$), in region *2* the gas density and velocity are determined by the boundary and initial conditions (the characteristic corresponding to $\xi_2 = u - c$, emerges from the straight line $x = v_1 t$).

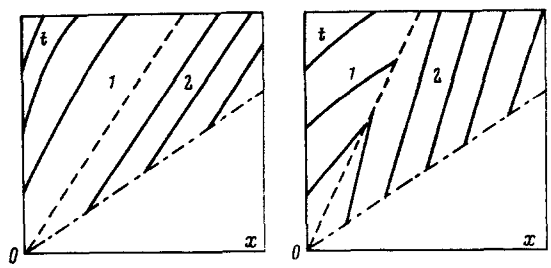

Fig.2

The solution of (1.5) in region *2* satisfying conditions (1.6) and (2.1) has the form

$$\varphi = \theta_1,\ f = \sigma\theta_1(\theta_1 - \theta),\ \theta_0 \leqslant \theta \leqslant \theta_1 \tag{2.2}$$

where θ_0 is the selfsimilar coordinate corresponding to the boundary between regions *1* and *2*.

Let us consider the case when $X(t)$ is a characteristic. The determinant of the system (1.5) composed of the coefficients of the derivatives f' and φ', vanishes at the point $\varphi = \theta + 1$. For a solution to exist, it is necessary that $f = \sigma\varphi$. The solution is continuous on the characteristic, therefore the gas flow velocity is $\varphi(\theta_0) = \theta_1$. The definition $X(t)$ $(dX/dt = \xi_2 = u - c)$ yields $\theta_0 = \theta_1 - 1$. The condition $f = \sigma\varphi$ at the point θ_0 holds, since the solution has the form

$$\varphi(\theta = \theta_0) = \theta_1,\ f(\theta = \theta_0) = \sigma\varphi(\theta_0) = \sigma\theta_1 \tag{2.3}$$

If the characteristics intersect on the line $X(t)$, then the solution has a discontinuity on $X(t)$. Let $D = dX/d$ The Hugoniot conditions for an isothermal gas can be written in selfsimilar variables in the form ($[F]$ denotes a jump in the function F)

$$[f(\varphi - \theta_0)] = 0,\ [f + f(\varphi - \theta_0)^2] = 0$$

whence

$$\varphi_1 = \theta_0 + (\varphi_2 - \theta_0)^{-1},\ f_1 = f_2(\varphi_2 - \theta_0)^2 \tag{2.4}$$

Here $f_1 = f(\theta_0 - 0)$, $\varphi_1 = \varphi(\theta_0 - 0)$ (the density and velocity of gas to the left of θ_0), $f_2 = f(\theta_0 + 0)$, $\varphi_2 = \varphi(\theta_0 + 0)$.

From (2.2) and (2.4) we obtain

$$\varphi_1 = \theta_0 + (\theta_1 - \theta_2)^{-1},\ f_1 = \sigma\theta_1(\theta_1 - \theta_0)^3 \tag{2.5}$$

The density f_1 and velocity of motion φ_1 are connected with θ_0 by the expression

$$f_1 = \sigma[\theta_0 + (\varphi_1 - \theta_0)^{-1}](\varphi_1 - \theta_0)^{-3} \tag{2.6}$$

When the solution is continuous, conditions (2.5) become (2.3).

3. Results and discussion. We will solve the problem (1.5)—(1.7) in region 1 by numerical methods. We introduce the mesh $\omega = \{\theta_n = nh; h = 0, 1, \ldots, N\}$ (h is a step along the coordinate) and mesh functions f_n and φ_n. In accordance with (1.7) we choose $\varphi_0 = \varphi(\theta = 0)$ and $f_n = f(\theta = 0) = 1$. Using the given values of φ_0 and f_0 we find f_n and φ_n at the points θ_n $(n = 1, 2, \ldots, N)$ using the Runge-Kutta fourth-order approximation method /9/. In the course

of the computation we find the coordinate θ_0 from condition (2.6), and θ_1 from (2.5).
In region 2 the solution is given by the formulas (2.2).

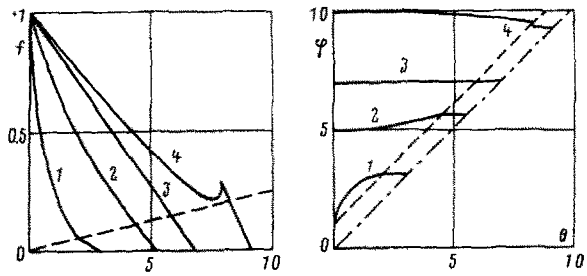

Fig.3

Fig.3 shows the relationships connecting $f(\theta)$ with $\varphi(\theta)$ for $\sigma = 0.02$ at $\varphi_0 = 1.1;\ 5;\ 7;\ 10$ (curves 1–4). When $\varphi_0 = \sigma^{-1/2}$ the solution is a simple wave (formula 1.8, curve 3). For $\varphi_0 < \sigma^{-1/2}$ the velocity φ increases as θ increases, attaining its maximum value at the point θ_0 which represents the coordinate of the weak discontinuity. If $\varphi_0 > \sigma^{-1/2}$, then φ decreases as θ increases. When $\theta = \theta_0$, the density f and gas flow velocity φ both change discontinuously. The solution is further described by (2.2). The dashed line in Fig.3 separates the solutions in regions 1 and 2. The dot-dash line $\varphi = \theta$ is the boundary of the region occupied by the gas.

The discontinuity appearing in the solution at $\theta = \theta_0\ (\varphi_0 > \sigma^{-1/2})$ is a shock wave moving in a direction opposite to the motion of the gas.

Just as in the case $\sigma = 0$, when $\sigma < 1$ the problem (1.5)–(1.7) has no subsonic solutions.

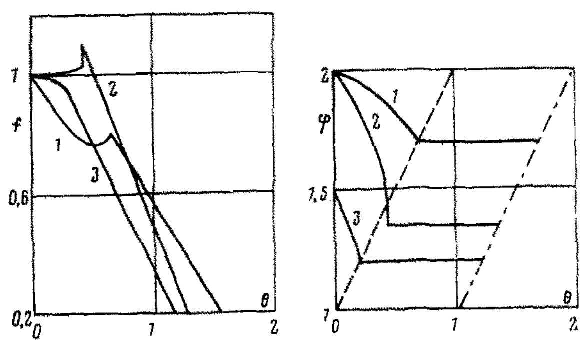

Fig.4

Fig.4 shows the result of computing the density $f(\theta)$ and velocity $\varphi(\theta)$ of the gas at $\sigma = 0.5;\ \varphi_0 = 2.0$ (curves 1), $\sigma = 0.95,\ \varphi_0 = 2.0$ (curves 2 and $\sigma = 0.95,\ \varphi_0 = 1.5$ (curves 3). We see that an increase in σ (φ_0 is fixed) leads to a more rapid decrease in φ as θ increases, with the jump in φ and f increased. In region 1 we observe a rise in density, leading to the appearance of a shock wave.

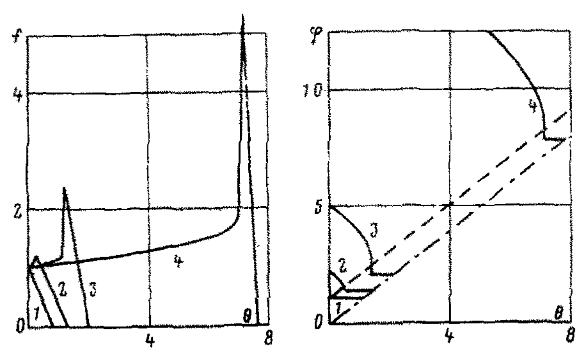

Fig.5

Fig.5 shows the results of solving problem (1.5)—(1.7) at $\sigma = 1$ for $\varphi_0 = 1; 1.5; 5; 20$ (curves 1—4). As φ_0 increases, the gas becomes more compressed and the value of the jump in the dynamic variables at the shock wave increases.

The motion of the gas at $\sigma > 1$ merits attention. Fig.6 shows the relations $f(\theta)$ and $\varphi(\theta)$ for $\sigma = 1.25$ at $\varphi_0 = 1.2; 1.06; 0.89$ (curves 1—3). When $\varphi_0 = \sigma^{1/4}$, condition (2.6) holds at the point $\theta = 0$. This means that when $\theta = 0$, φ and f change their values discontinuously (a shock wave is situated at the boundary of the medium). The solution behind the shock wave is subsonic and is described by (2.2). When $\varphi_0 < \sigma^{1/4}$, there are no supersonic gas flows.

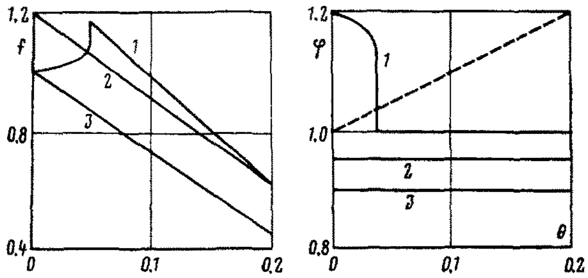

Fig.6

When $\sigma > 1$, a subsonic flow exists unlike when $0 \leqslant \sigma \leqslant 1$, identical with the solution (1.8) of the filtration problem.

Thus system (1.1) with conditions (1.2), (1.3) admits not only of the filtration solution (1.8), but also a number of other solutions depending on the rate of inflow of gas u_0. If $a < A$, we have only a supersonic gas flow, and a shock wave forms when $u_0 > u^*_0 = c\sqrt{A/a}$ moving in a direction opposite to that of the gas flow. An increase in a (or decrease in A) leads to a decrease in u_0^*, and an increase in the jumps in dynamic variables (the density and velocity) of the gas on the shock wave.

In the case when $a > A$ supersonic inflow of gas is possible only when $u_0 > c(a/A)^{1/4}$, and a shock wave always forms. Moreover, when $a > A$, a subsonic gas inflow is possible, identical with the solution of the filtration problem.

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